Section 17.3

The Divergence Theorem

The Divergence Theorem

- Examples, Computing Closed Surface Integrals
- Examples, Surface Integrals using Alternative Surfaces

1 The Divergence Theorem

The Divergence Theorem

Let $\mathcal{S}$ be a closed surface that encloses a solid $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing outside $\mathcal{W}$. Let $\vec{\mathsf{F}}$ be a vector field whose domain contains $\mathcal{W}$. Then:

$$\oiint_{\mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = \iiint_{\mathcal{W}} \operatorname{div}(\vec{\mathsf{F}})\, dV.$$

- Analogy: Adding up the amount of stuff supplied $(+)$ or consumed $(-)$ by all nodes in a network equals the total amount of stuff supplied to $(+)$ or consumed from $(-)$ outside the network.
- The left-hand side measures the total flux outward through $\mathcal{S}$ — that is, the amount of stuff supplied outside $\mathcal{W}$.
- The right-hand side is the integral over $\mathcal{W}$ of the amount of stuff supplied by each point.

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The Divergence Theorem

Example 1: Find the flux of the vector field $\vec{\mathsf{F}}(x,y,z) = \langle z, y, x \rangle$ out the unit sphere $\mathcal{S}$ defined by $x^2 + y^2 + z^2 = 1$.

<u>Solution:</u> Let $\mathcal{W}$ be the unit ball, so that $\mathcal{S} = \partial\mathcal{W}$. Here $\text{div}(\vec{\mathsf{F}}) = 1$, so by the Divergence Theorem,

$$\oiint_{\mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = \iiint_{\mathcal{W}} \text{div}(\vec{\mathsf{F}})\, dV = \text{volume}(\mathcal{W}) = \frac{4\pi}{3}.$$

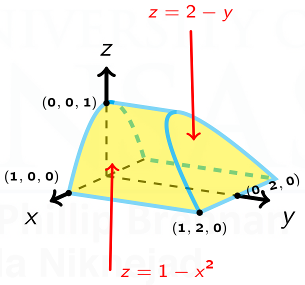
The Divergence Theorem and Volume: If $\text{div}(\vec{\mathsf{F}})$ is a constant c, then

$$\oiint_{\partial\mathcal{W}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = c(\text{volume}(\mathcal{W}))$$

for **any** solid region $\mathcal{W}$ (orienting its boundary with outward normals).

Example 2: Let $\vec{F}(x, y, z) = \left\langle xy, y^2 + e^{xz^2}, \sin(xy) \right\rangle$, and let $\mathcal{S}$ be the surface of the solid $\mathcal{W}$ bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, and $y + z = 2$. Evaluate $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

Solution: Parametrizing $\mathcal{S}$ would be hard, but we can use the Divergence Theorem.



The solid $\mathcal{W}$ consists of points (x, y, z) such that

- $x \in [-1, 1]$;
- $0 \leq y \leq 2 - z$;
- $0 \leq z \leq 1 - x^2$.

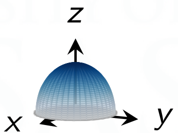
Video

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{W}} \operatorname{div}(\vec{F})\, dV = \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} 3y\, dy\, dz\, dx = \frac{184}{35}$$

Changing the Surface of Integration

Example 3: A compressible (opposite of incompressible) fluid is flowing through a net described by the equation $\mathcal{S}: z = \sqrt{9 - x^2 - y^2}$ and oriented in the positive z-direction. Determine the flow rate of the fluid across the net if the velocity vector field for the fluid is given by and $\vec{F} = \langle \sin(y^2), \ln(x^2+1), z \rangle$. (Find $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.)

Solution: The surface is the upper hemisphere of radius 3; the surface integral is computationally complicated if directly computed.
Alternative surface: We close the upper hemisphere with a disk of radius 3 on xy-plane:
$\mathcal{S}_{\text{new}} : \langle r\cos(t), r\sin(t), 0 \rangle$ for $0 \le r \le 3$ and $0 \le t \le 2\pi$.



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▸ Link-Surfaces

The triple integral of the divergence of $\vec{F}$ over the solid entrapped between the two surfaces $\iiint_{\mathcal{W}} \text{Div}(\vec{F})\, dv$ is equal to the outward flux:

$$\underbrace{\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}}_{\text{Surface we want to find}} + \underbrace{\iint_{\mathcal{S}_{\text{new}}} \vec{F} \cdot d\vec{S}}_{\text{Alternative Surface}} = \underbrace{\iiint_{\mathcal{W}} \overbrace{\text{Div}(\vec{F})}^{=\frac{\partial \sin(y^2)}{\partial x} + \frac{\partial \ln(x^2+1)}{\partial y} + \frac{\partial z}{\partial z} = 1} dv}_{\iiint_{\mathcal{W}} 1\, dv}$$

Example 3 (continued)

Compute the alternative surface integral minding that unit outward normal to the disk is $-\vec{k}$:

$$\iint_{\mathcal{S}_{\textbf{new}}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = \iint_{\mathcal{D}_{\textbf{new}}} \underbrace{\langle \text{something}, \text{something}, 0 \rangle}_{\vec{\mathsf{F}}} \cdot \underbrace{\langle 0, 0, -r \rangle}_{-r\vec{k}} dA = 0$$

Compute the triple integral: $$\iiint_{\mathcal{W}} \text{Div}(\vec{\mathsf{F}})\, dv = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \underbrace{\rho^2 \sin(\phi)}_{\text{Jac}} d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \left. \frac{\rho^3}{3} \right|_0^3 \sin(\phi) d\phi\, d\theta$$

$$= 9 \int_0^{2\pi} \left. -\cos(\phi) \right|_0^{\frac{\pi}{2}} d\theta = 18\pi$$

By Divergence Theorem: $$\underbrace{\iint_{\mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}}}_{\text{We want to find}} + \underbrace{\iint_{\mathcal{S}_{\textbf{new}}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}}}_{=0} = \underbrace{\iiint_{\mathcal{W}} \text{Div}(\vec{\mathsf{F}})\, dv}_{=18\pi}$$

So $\iint_{\mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = 18\pi$

Note that $\iint_{\mathcal{W}} 1\, dv$ is equal to half the volume of a sphere with radius 3 and you don't need to compute it.

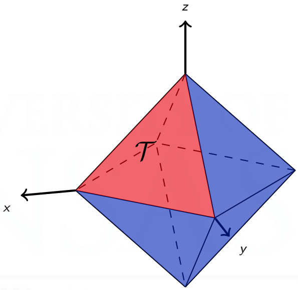
Changing the Surface of Integration

Example 4 (The original was provided by Prof. Martin): Define $\mathcal{W}, \mathcal{T}, \mathcal{S}, \vec{F}$ as follows:

$\mathcal{W}$ = octahedron with vertices at $\pm\vec{i}, \pm\vec{j}, \pm\vec{k}$
$\mathcal{T}$ = triangle with vertices $\vec{i}, \vec{j}, \vec{k}$
$\mathcal{S}$ = surface consisting of the other seven faces of $\mathcal{W}$, all oriented outwards

$\vec{F}(x,y,z) = \langle 3x+2y,\ x-2y+2z,\ -x+4z \rangle$

Find $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.



Solution: Integrating over $\mathcal{S}$ directly would be tiresome. Instead, use the Divergence Theorem:

$$\iiint_{\mathcal{W}} \nabla \cdot \vec{F}\, dV = \oiint_{\partial\mathcal{W}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} + \iint_{\mathcal{T}} \vec{F} \cdot d\vec{S}$$

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{W}} \nabla \cdot \vec{F}\, dV - \iint_{\mathcal{T}} \vec{F} \cdot d\vec{S} \qquad (*)$$

Changing the Surface of Integration

Example 4 (continued): First, calculate

$$\iiint_{\mathcal{W}} \nabla \cdot \vec{\mathsf{F}}\, dV \;=\; \iiint_{\mathcal{W}} 5\, dV \;=\; 5 \cdot \text{volume}(\mathcal{W}) \;=\; 5 \cdot \frac{8}{6} \;=\; \frac{20}{3}$$

since slicing along the coordinate planes partitions $\mathcal{W}$ into eight tetrahedra, each of base $1/2$ and height 1, hence volume $1/6$.

Second, $\mathcal{T}$ has normal vector $\langle 1, 1, 1\rangle$ and is the graph of $z = 1 - x - y$ over the triangle in $\mathbb{R}^2$ bounded by $x = 0$, $y = 0$, $x + y = 1$, so

$$\begin{aligned}\iint_{\mathcal{T}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} \;&=\; \int_0^1 \int_0^{1-y} \vec{\mathsf{F}}(x, y, 2 - x - y) \cdot \langle 1, 1, 1\rangle\, dy\, dx \\ &=\; \int_0^1 \int_0^{1-y} (-2x - 6y + 5)\, dx\, dy \;=\; \frac{11}{2}.\end{aligned}$$

So equation $(*)$ gives

$$\iint_{\mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} \;=\; \frac{20}{3} - \frac{11}{2} \;=\; \boxed{\frac{7}{6}}$$